On Weak values and the Reconstruction Problem

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TSQM and Weak Values

In time-symmetric quantum mechanics (TSQM) the state of a system is represented by a two-state vector $\langle\Phi|\ |\Psi\rangle$ where the state $\langle\Phi|$ evolves backwards from the future and the state $|\Psi\rangle$ evolves forwards from the past. Choose t be such that: $t_{\mathrm{i}} < t < t_{\mathrm{f}}$. Following the time-symmetric approach to quantum mechanics at this intermediate time the system is described by the *two* wavefunctions

$$\Psi = U_{\mathrm{i}}(t, t_{\mathrm{i}})\Psi(t_{\mathrm{i}}) \quad , \quad \Phi = U_{\mathrm{f}}(t, t_{\mathrm{f}})\Phi(t_{\mathrm{f}})$$

where $U_{\mathrm{i}}(t, t') = e^{-i\widehat{H_{\mathrm{i}}}(t-t')/\hbar}$ and $U_{\mathrm{f}}(t, t') = e^{-i\widehat{H_{\mathrm{f}}}(t-t')/\hbar}$ are the unitary operators governing the evolution of the state before and after time t. The weak value of a quantum observable $\widehat{A}$ is by definition

$$\langle\widehat{A}\rangle_{\Phi,\Psi} = \frac{\langle\Phi|\widehat{A}|\Psi\rangle}{\langle\Phi|\Psi\rangle} \quad , \quad \langle\Phi|\Psi\rangle \neq 0.$$

Wigner Distribution

The states $|\Psi\rangle$ and $|\Phi\rangle$ can be described by their Wigner distributions

$$
\begin{aligned}
W_\Psi(x,p) &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\Psi(x+\tfrac{1}{2}y)\Psi^*(x-\tfrac{1}{2}y)d^ny \\
W_\Phi(x,p) &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\Phi(x+\tfrac{1}{2}y)\Phi^*(x-\tfrac{1}{2}y)d^ny.
\end{aligned}
$$

These are would-be probability distributions (they usually take values < 0), but they have the right marginals:

$$
\begin{aligned}
\int W_\Psi(x,p)d^np &= |\Psi(x)|^2 \\
\int W_\Psi(x,p)d^nx &= |\widehat{\Psi}(p)|^2.
\end{aligned}
$$

If $\Psi \in L^2(\mathbb{R}^n)$ then $W_\Psi \in L^2(\mathbb{R}^{2n})$.

But what is the WD?

Let $\widehat{\Pi}$ be the parity operator: $\widehat{\Pi}\Psi(x) = \Psi(-x)$ that is $\langle x|\widehat{\Pi}|\Psi\rangle = \langle -x|\Psi\rangle$. Let $\widehat{T}_{x,p} = e^{\frac{i}{\hbar}(p\widehat{x} - x\widehat{p})}$ be the displacement operator, and define the reflection operator about (x_0, p_0) by

$$\widehat{\Pi}_{x,p} = \widehat{T}_{x,p}\widehat{\Pi}\widehat{T}_{x,}^{\dagger}$$

("Grossmann–Royer operator"). Then we can rewrite the Wigner distribution as

$$W_\Psi(x,p) = \left(\tfrac{1}{\pi\hbar}\right)^n \langle\Psi|\widehat{\Pi}_{x,p}|\Psi\rangle.$$

Thus $W_\Psi(x,p)$ is proportional to the overlap of Ψ with its mirror image about (x,p); which is a measure of how much Ψ is "centered" about the point (x,p).

At time t the state is represented by the wavefunction $\Phi + \Psi$; its Wigner distribution is thus

$$W_{\Phi+\Psi} = W_{\Phi} + W_{\Psi} + 2\,\mathrm{Re}\, W_{\Psi,\Phi}$$

where $W_{\Psi,\Phi} = \left(\frac{1}{\pi\hbar}\right)^n \langle\Phi|\widehat{\Pi}_{x,p}|\Psi\rangle$. Explicitly:

$$W_{\Psi,\Phi}(x,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\Psi(x+\tfrac{1}{2}y)\Phi^*(x-\tfrac{1}{2}y)d^n y$$

is the *cross-Wigner transform* of the two-state vector $\langle\Phi|\ |\Psi\rangle$. It describes a **strong interference effect**.

One could as well consider the cross-ambiguity transform

$$A_{\Psi,\Phi}(x,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\Psi(y+\tfrac{1}{2}x)\Phi^*(y-\tfrac{1}{2}x)d^n y$$

which signals a **strong correlation** between the states $|\Psi\rangle$ and $|\Phi\rangle$.

Cross-Wigner and Weak Values

Let $a(x,p)$ be the classical observable whose quantization (in the Weyl scheme) is $\widehat{A}$:

$$\widehat{A} = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint \widehat{a}(x,p) e^{\frac{i}{\hbar}(x\widehat{x}+p\widehat{p})} d^n p d^n x.$$

Moyal showed in 1949 that

$$\langle \widehat{A} \rangle_\Psi = \iint a(x,p) W_\Psi(x,p) d^n p d^n x.$$

The number $\langle \widehat{A} \rangle_\Psi$ is thus the average value of the classical observable weighted by Wigner's distribution: probabilistic phase space interpretation of quantum mechanics.

Generalized Moyal Formula

One shows, using some "simple" mathematics, that

$$\langle \Phi | \widehat{A} | \Psi \rangle = \iint a(x,p) W_{\Psi,\Phi}(x,p) d^n p d^n x$$

and hence the weak value $\langle \widehat{A} \rangle_{\Phi,\Psi}$ can be expressed as

$$\langle \widehat{A} \rangle_{\Phi,\Psi} = \frac{1}{\langle \Phi | \Psi \rangle} \iint a(x,p) W_{\Psi,\Phi}(x,p) d^n p d^n x.$$

We have the generalized marginal properties

$$\int W_{\Psi,\Phi}(x,p) d^n p = \Psi(x) \Phi^*(x)$$
$$\int W_{\Psi,\Phi}(x,p) d^n x = \Psi(p) \widehat{\Phi}^*(p).$$

It follows that

$$\rho_{\Phi,\Psi}(x,p) = \frac{W_{\Psi,\Phi}(x,p)}{\langle \Phi | \Psi \rangle} \tag{1}$$

is a complex probability distribution.

An Example

Let $\Phi(x) = \Theta(x - x_0)$, $\Psi(x) = \Theta(x + x_0)$ where

$$\Theta(x) = (\pi\hbar)^{-1/4} e^{-x^2/2\hbar}$$

(Gaussian cat-state at time t). We have

$$W_{\Psi,\Phi}(x,p) = \frac{1}{\pi\hbar} e^{\frac{i}{\hbar}px_0} e^{-\frac{1}{\hbar}(x^2+p^2)} \quad , \quad \langle\Phi|\Psi\rangle = e^{-\frac{1}{\hbar}x_0^2}.$$

Choose $a(x,p) = \pi\hbar\delta(x,p)$. Then $\widehat{A}$ is the parity operator $\widehat{\Pi}\Psi(x) = \Psi(-x)$ and we have

$$\langle\widehat{P}\rangle_{\Phi,\Psi} = e^{\frac{i}{\hbar}px_0} e^{\frac{1}{\hbar}x_0^2}.$$

The eigenvalues of $\widehat{P}$ are ± 1, but the weak value can take **arbitrary large values**; as $a \to \infty$ the preselected and post-selected states become orthogonal and $\langle\widehat{P}\rangle_{\Phi,\Psi} \to \infty$.

Reconstruction problem

In Nature, **474**(7350), (2011), Lundeen *et al.* show that one can reconstruct Ψ by performing a weak measurement of position followed by a strong measurement of momentum yielding a value p_0. They prove that

$$\langle \widehat{\pi_{x_0}} \rangle_{\Phi,\Psi} = \frac{e^{-\frac{i}{\hbar}p_0 x_0}\Psi(x_0)}{\sqrt{2\pi\hbar}\widehat{\Psi}(p_0)}$$

Let $\pi_{x_0}(x) = \delta(x - x_0)$; using Moyal's formula we have

$$\begin{aligned}
\langle \widehat{\pi_{x_0}} \rangle_{\Phi,\Psi} &= \frac{1}{\langle \Phi|\Psi \rangle} \iint \delta(x - x_0) W_{\Psi,\Phi}(x,p) dp dx \\
&= \frac{1}{\langle \Phi|\Psi \rangle} \int_{-\infty}^{\infty} W_{\Psi,\Phi}(x_0,p) dp \\
&= \frac{\Psi(x_0)\Phi^*(x_0)}{\langle \Phi|\Psi \rangle}
\end{aligned}$$

hence Lundeen's formula since $\langle \Phi|\Psi \rangle = \langle p_0|\Psi \rangle = \widehat{\Psi}(p_0)$ and $\Phi(x_0) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p_0 x_0}$.

Reconstructing the two-state vector

Both states $|\Psi\rangle$ and $|\Phi\rangle$ are uniquely determined by the interference term $W_{\Phi,\Psi}(x,p)$ (or equivalently, the correlation term $A_{\Phi,\Psi}(x,p)$). In fact, choose an arbitrary auxiliary state Λ such that $\langle\Phi|\Lambda\rangle \neq 0$; then

$$\Psi(x) = \frac{2^n}{\langle\Phi|\Lambda\rangle} \iint e^{\frac{2i}{\hbar}p(x-y)} W_{\Psi,\Phi}(y,p)\Lambda(2y-x)d^n p d^n y$$

and, if $\langle\Psi|\Lambda\rangle \neq 0$, noting that $W_{\Phi,\Psi} = W^*_{\Psi,\Phi}$,

$$\Phi(x) = \frac{2^n}{\langle\Psi|\Lambda\rangle} \iint e^{\frac{2i}{\hbar}p(x-y)} W^*_{\Psi,\Phi}(y,p)\Lambda(2y-x)d^n p d^n y.$$

The knowledge of the interference term $W_{\Phi,\Psi}$ at time t uniquely determines both the pre- and post-selected states!

The Born–Jordan Case

We have the Weyl monomial quantization rule

$$\mathrm{Weyl}: x^r p^s \longrightarrow \frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}\widehat{p}^{s-k}\widehat{x}^r\widehat{p}^k$$

whereas in the Born–Jordan case

$$\mathrm{BJ}: x^r p^s \longrightarrow \frac{1}{s+1}\sum_{k=0}^{s}\widehat{p}^{s-k}\widehat{x}^r\widehat{p}^k.$$

They yield different results as soon as $r, s \geq 2$.
The Born–Jordan was the first formal quantization rule ever proposed. It is probably the correct one (otherwise matrix mechanics is inconsistent!). Open question...

The Born–Jordan Case

In the general case we have

$$\widehat{A}_{\mathrm{Weyl}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint \widehat{a}(x,p) e^{\frac{i}{\hbar}(x\widehat{x}+p\widehat{p})} d^n p d^n x.$$

for the Weyl quantization, and

$$\widehat{A}_{\mathrm{BJ}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint \widehat{a}(x,p) e^{\frac{i}{\hbar}(x\widehat{x}+p\widehat{p})} \left(\frac{\sin(px/2\hbar)}{px/2\hbar}\right) d^n p d^n x$$

in Born–Jordan quantization. Both quantizations are *identical* for Hamiltonians $T+V$, and also when a vector potential is present. They are different for the square of the angular momentum ("angular momentum dilemma").

The reconstruction formula is much more problematic to establish when one uses the Born–Jordan quantization of an observable.

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